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## Problems I: Mathematical Statements and Proofs

1. By using truth tables prove that, for all statements $P$ and $Q$, the statement $'P \rightarrow Q'$ and its contrapositive $'(not\ Q) \rightarrow (not\ P)'$ are equivalent. In example 1.2.3 identify which statement is the contrapositive of statement (i) $(f(a) = 0 \rightarrow a > 0)$. Find another pair of statements in that list that are the contrapositives of each other.

   Truth table

   | $P$ | $Q$ | $\sim P$ | $\sim Q$ | $P \rightarrow Q$ | $\sim Q \rightarrow \sim P$ |
   |---|---|---|---|---|---|
   | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
   | $T$ | $F$ | $F$ | $T$ | $F$ | $F$ |
   | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
   | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

   Since the last two columns are identical, the statements $'P \rightarrow Q'$ and its contrapositive $'\sim Q \rightarrow \sim P'$ are logically equivalent.
   The contrapositive of statement (i) $(f(a) = 0 \rightarrow a > 0)$ is statement (vii) $(a \leq 0 \rightarrow f(a) \neq 0)$. Similarly, the contrapositive of statement (iii) $(f(a) = 0 \rightarrow a \leq 0)$ is statement (vi) $(a > 0 \rightarrow f(a) \neq 0)$.

2. By using truth tables prove that, for all statements $P$ and $Q$, the three statements (i) $'P \rightarrow Q'$, (ii) $'(P\ or\ Q) \leftrightarrow Q'$, and (iii) $'(P\ and\ Q) \leftrightarrow P'$ are equivalent.

   Truth table

   | $P$ | $Q$ | $P\ or\ Q$ | $P\ and\ Q$ | $P \rightarrow Q$ | $(P\ or\ Q) \leftrightarrow Q$ | $(P\ and\ Q) \leftrightarrow P$ |
   |---|---|---|---|---|---|---|
   | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
   | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ |
   | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |
   | $F$ | $F$ | $F$ | $F$ | $T$ | $T$ | $T$ |

   Since the last three columns are identical, the statements $'P \rightarrow Q'$, $'(P\ or\ Q) \leftrightarrow Q'$, and $'(P\ and\ Q) \leftrightarrow P'$ are logically equivalent.

3. Prove that the three basic connectives 'or', 'and', and 'not' can all be written in terms of the single connective 'notand' where $'P\ notand\ Q'$ is interpreted as $'not(P\ and\ Q)'$.

   | $P$ | $P\ and\ P$ | $\sim(P\ and\ P)$ | $\sim P$ |
   |---|---|---|---|
   | $T$ | $T$ | $F$ | $F$ |
   | $F$ | $F$ | $T$ | $T$ |

   Since the last two columns are identical, the statements $\sim P$ and $'\sim(P\ and\ P)'$ are logically equivalent. Hence we can write $\sim P$ as $'P\ notand\ P'$.

   | $P$ | $Q$ | $P\ and\ Q$ | $\sim(P\ and\ Q)$ | $\sim[\sim(P\ and\ Q)]$ |
   |---|---|---|---|---|
   | $T$ | $T$ | $T$ | $F$ | $T$ |
   | $T$ | $F$ | $F$ | $T$ | $F$ |
   | $F$ | $T$ | $F$ | $T$ | $F$ |
   | $F$ | $F$ | $F$ | $T$ | $F$ |

Since columns 3 and 5 are identical, the statements $'P \textit{ and } Q'$ and $'{\sim}[{\sim}(P \textit{ and } Q)]'$ are logically equivalent. But observe that, by definition, $'{\sim}(P \textit{ and } Q)'$ is written as $'P \textit{ notand } Q'$. Thus $'{\sim}[{\sim}(P \textit{ and } Q)]'$ is logically equivalent to $'{\sim}(P \textit{ notand } Q)'$. Then, by the first part of the problem, $'{\sim}(P \textit{ notand } Q)'$ can be written as $'(P \textit{ notand } Q) \textit{ notand } (P \textit{ notand } Q)'$. Hence we can write $'P \textit{ and } Q'$ as $'(P \textit{ notand } Q) \textit{ notand } (P \textit{ notand } Q)'$.

| $P$ | $Q$ | $P$ or $Q$ | $\sim P$ | $\sim Q$ | $\sim P$ and $\sim Q$ | $\sim(\sim P$ and $\sim Q)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $F$ |

Since columns 3 and 7 are identical, the statements $'P \textit{ or } Q'$ and $'{\sim}({\sim}P \textit{ and } {\sim}Q)'$ are logically equivalent. But observe that, by the first part of the problem, $'{\sim}P \textit{ and } {\sim}Q'$ is logically equivalent to $'(P \textit{ notand } P) \textit{ and } (Q \textit{ notand } Q)'$. Now, by definition, the negation of this last statement, namely $'{\sim}({\sim}P \textit{ and } {\sim}Q)'$, can be written as $'(P \textit{ notand } P) \textit{ notand } (Q \textit{ notand } Q)'$. Thus we can write $'P \textit{ or } Q'$ as $'(P \textit{ notand } P) \textit{ notand } (Q \textit{ notand } Q)'$.

4. Prove the following statements concerning the positive integers $a$, $b$, and $c$.
(i) ($a$ divides $b$) and ($a$ divides $c$) $\rightarrow$ $a$ divides $(b + c)$.
(ii) ($a$ divides $b$) or ($a$ divides $c$) $\rightarrow$ $a$ divides $bc$.

(i) $a$ divides $b$ means $b = aq$ for some integer $q$, and $a$ divides $c$ means $c = ap$ for some integer $p$. Thus $b + c = aq + ap = a(q + p) = ak$, where $k = q + p$ is an integer. Therefore $a$ divides $(b + c)$.

(ii) <u>case 1:</u> $a$ divides $b$ means $b = aq$ for some integer $q$. Thus $bc = (aq)c = a(qc) = ar$, where $r = qc$ is an integer. Therefore $a$ divides $bc$.
<u>case 2:</u> $a$ divides $c$ means $c = ap$ for some integer $p$. Thus $bc = b(ap) = a(bp) = as$, where $s = bp$ is an integer. Therefore $a$ divides $bc$.
Hence, in either case, $a$ divides $bc$.

5. Which of the following conditions are necessary for the positive integer $n$ to be divisible by 6 (proofs not necessary)?
(i) 3 divides $n$.
(ii) 9 divides $n$.
(iii) 12 divides $n$.
(iv) $n = 12$.
(v) 6 divides $n^2$.
(vi) 2 divides $n$ and 3 divides $n$.
(vii) 2 divides $n$ or 3 divides $n$.
Which of these conditions are sufficient?

6 divides $n$ means $n = 6q$ for some integer $q$. The following conditions are necessary for the positive integer $n$ to be divisible by 6: (i) 3 divides $n$, (v) 6 divides $n^2$, (vi) 2 divides $n$ and 3 divides $n$, and (vii) 2 divides $n$ or 3 divides $n$. The following conditions are sufficient for the positive integer $n$ to be divisible by 6: (iii) 12 divides $n$, (iv) $n = 12$, (v) 6 divides $n^2$, and (vi) 2 divides $n$ and 3 divides $n$.

6. Use the properties of addition and multiplication of real numbers given in Properties 2.3.1 to deduce that, for all real numbers $a$ and $c$,
   (i) $a \times 0 = 0 = 0 \times a$,
   (ii) $(-a)b = -ab = a(-b)$,
   (iii) $(-a)(-b) = ab$.

   (i) Prove that $a \times 0 = 0 = 0 \times a$.

   | | |
   |---|---|
   | $0 + 0 = 0$ | Additive identity (iv) |
   | $(0 \times a) + (0 \times a) = 0 \times a$ | Distributive property (iii) |
   | $0 \times a = 0$ | Additive inverse (vi) |
   | $a \times 0 = 0$ | Commutative property (i) |

   Thus $a \times 0 = 0 = 0 \times a$.

   (ii) Prove that $(-a)b = -ab = a(-b)$.
   First we show that $-a = (-1)a$ by showing that $a + (-1)a = 0$.

   | | |
   |---|---|
   | $a + (-1)a = 1a + (-1)a$ | Multiplicative Identity (v) |
   | $= (1 + (-1))a$ | Distributive property (iii) |
   | $= 0a$ | Since $-1$ is the additive inverse of 1 |
   | $= 0$ | By part (i) of problem |

   Thus $a + (-1)a = 0 \ \rightarrow\ -a = (-1)a$. Then

   | | |
   |---|---|
   | $-ab = (-1)ab$ | By the proof above |
   | $= ((-1)a)b$ | Associative property (ii) |
   | $= (-a)b$ | By the proof above |
   | $-ab = (-1)ab$ | By the proof above |
   | $= ((-1)b)a$ | Associative property (ii) |
   | $= (-b)a$ | By the proof above |
   | $= a(-b)$ | Commutative property (i) |

   Thus $(-a)b = -ab = a(-b)$.

   (iii) Prove that $(-a)(-b) = ab$.

   | | |
   |---|---|
   | $a + (-a) = 0$ | $= (-a)(-b)$ |
   | $a = -(-a)$ | Additive Identity (iv) |
   | $ab = -(-a)b$ | |
   | $= -(-ab)$ | Since $a = -(-a)$ |
   | $= -\big((-a)b\big)$ | By part (ii) of problem |
   | $= -\big(b(-a)\big)$ | By part (ii) of problem |
   | $= (-b)(-a)$ | Commutative property (i) |

By part (ii) of problem Commutative property (i)

Thus $(-a)(-b) = ab$.

7. Prove by contradiction the following statement concerning an integer $n$.

$$n^2 \text{ is even} \rightarrow n \text{ is even.}$$

[You may suppose that an integer $n$ is odd if and only if $n = 2q + 1$ for some integer $q$. This is proved later as Proposition 11.3.4.]

Suppose $n$ is not even. Then $n$ is odd, that is $n = 2q + 1$ for some integer $q$. Thus $n^2 = (2q + 1)^2 = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1 = 2p + 1$ where $p = 2q^2 + 2q$ is an integer. Thus $n^2$ is odd contradicting that $n^2$ is even. It follows that our initial assumption, that $n$ is odd, is false. Hence $n$ is even as required. Therefore, $n^2$ is even $\rightarrow n$ is even.

8. Prove the following statements concerning a real number $x$.

(i) $x^2 - x - 2 = 0 \leftrightarrow x = -1$ or $x = 2$.

(ii) $x^2 - x - 2 > 0 \leftrightarrow x < -1$ or $x > 2$.

(i) $' \rightarrow '$: $x^2 - x - 2 = 0 \rightarrow (x - 2)(x + 1) = 0$

$\rightarrow (x - 2) = 0$ or $(x + 1) = 0$

$\rightarrow x = 2$ or $x = -1$

Thus $x^2 - x - 2 = 0 \rightarrow x = -1$ or $x = 2$.

$' \leftarrow '$: If $x = 2$, then $x^2 - x - 2 = 2^2 - 2 - 2 = 0$. If $x = -1$, then $x^2 - x - 2 = (-1)^2 - (-1) - 2 = 0$. So, in either case, $x^2 - x - 2 = 0$. Thus ($x = -1$ or $= 2$) $\rightarrow x^2 - x - 2 = 0$.

Hence $x^2 - x - 2 = 0 \leftrightarrow x = -1$ or $x = 2$.

(ii) $' \rightarrow '$: $x^2 - x - 2 > 0 \rightarrow (x - 2)(x + 1) > 0$

$\rightarrow (x - 2 > 0$ and $x + 1 > 0)$ or $(x - 2 < 0$ and $x + 1 < 0)$

$\rightarrow (x > 2$ and $x > -1)$ or $(x < 2$ and $x < -1)$

$\rightarrow x > 2$ or $x < -1$

Thus $x^2 - x - 2 > 0 \rightarrow x < -1$ or $x > 2$.

$' \leftarrow '$:

<u>case1:</u> $x > 2 \rightarrow 2x > 4 > 2$ (multiply by $2 > 0$) $\rightarrow 2x > 2$ and $x > 2 \rightarrow x^2 > 2x > 2$ (multiply by $x > 0$) $\rightarrow x^2 > 2$. It follows that $x^2 - x > 2x - x = x > 2$ and so $x^2 - x > 2 \rightarrow x^2 - x - 2 > 0$ as required.

<u>case2:</u> $x < -1 \rightarrow 0 < -1 - x \rightarrow 0 < 2 < 1 - x$ (by adding 2) $\rightarrow 0 < 1 - x$ and $x < -1 \rightarrow -x > 1$ (multiply by $-1 < 0$) and $x < -1 \rightarrow x^2 > -x > 1$ (multiply by $x < 0$) $\rightarrow x^2 > 1 \rightarrow x^2 - x > 1 - x > 0 \rightarrow x^2 - x - 2 > 1 - x - 2 = -1 - x > 0 \rightarrow x^2 - x - 2 > 0$ as required.

Hence $x^2 - x - 2 > 0 \leftrightarrow x < -1$ or $x > 2$.

9. Prove by contradiction that there does not exist a largest integer.

[Hint: Observe that for any integer $n$ there is a greater one, say $n + 1$. So begin your proof
Suppose for contradiction that there is a largest integer. Let this larger integer be $n$. ...]

Suppose for contradiction that there is a largest integer $n$. Observe that $0 < 1 \to n < n + 1$. Thus $n$ is not the largest integer, since for all $n$, $n + 1 > n$.

10. What is wrong with the following proof that 1 is the largest integer?
    Let $n$ be the largest integer. Then, since 1 is an integer we must have $1 \leq n$. On the other hand, since $n^2$ is also an integer we must have $n^2 \leq n$ from which it follows that $n \leq 1$. Thus, since $1 \leq n$ and $n \leq 1$ we must have $n = 1$. Thus 1 is the largest integer as claimed.
    What does this argument prove?

    The proof starts with a statement which is false (from problem 9). We also know that the conclusion is false since 1 is not the largest integer. However all the implications that start with a false hypothesis are true. In fact, this argument proves that if a largest integer existed, it would be 1.

11. Prove by contradiction that there does not exist a smallest positive real number.

    Suppose for contradiction that $n$ is the smallest positive real number. Observe that $0 < \frac{1}{2} < 1 \to 0 < \frac{1}{2}n < n$. Thus the number $\frac{1}{2}n$ is positive, real, and less than $n$, contradicting our initial assumption that $n$ was the smallest positive real number. Hence there does not exist a smallest positive real number.

12. Prove by induction on $n$ that, for all positive integers $n$, 3 divides $4^n + 5$.

    3 divides $4^n + 5$ means $4^n + 5 = 3q$ for some integer $q$.
    <u>Base case:</u> For $n = 1$, $4^n + 5 = 4^1 + 5 = 9$ which is divisible by 3 as required.
    <u>Inductive step:</u> Suppose now as inductive hypothesis that, for some positive integer $k$, 3 divides $4^k + 5$, that is $4^k + 5 = 3q$ for some integer $q$. We need to show that 3 divides $4^{k+1} + 5$, that is $4^{k+1} + 5 = 3q$ for some integer $q$. Then $4^{k+1} + 5 = 4 \cdot 4^k + 5$ (by inductive definition) $= 4(3q - 5) + 5$ (by inductive hypothesis) $= 12q - 15 = 3(4q - 5) = 3p$, where $p = 4q - 5$ is an integer. Thus 3 divides $4^{k+1} + 5$ as required.
    <u>Conclusion:</u> Hence, by induction on $n$, 3 divides $4^n + 5$ for all positive integers $n$.

13. Prove by induction on $n$ that $n! > 2^n$ for all integers $n$ such that $n \geq 4$.

    <u>Base case:</u> For $n = 4$, $n! = 4! = 24 > 16 = 2^4 = 2^n$ as required.
    <u>Inductive step:</u> Suppose now as inductive hypothesis that $k! > 2^k$ for some positive integer $k \geq 4$. We need to show that $(k + 1)! > 2^{k+1}$. Then $2k! >$

$2 \cdot 2^k = 2^{k+1}$(by inductive hypothesis) and $(k+1)! = (k+1)k!$ (by inductive definition) $\geq (4+1)k! = 5k!$ (since $k \geq 4$) $> 2k! > 2 \cdot 2^k = 2^{k+1}$ (by inductive hypothesis). Thus $(k+1)! > 2^{k+1}$ as required.
Conclusion: Hence, by induction on $n$, $n! > 2^n$ for all integers $n \geq 4$.

14. Prove Bernoulli's inequality
$$(1+x)^n \geq 1 + nx$$
for all non-negative integers $n$ and real numbers $x > -1$.

Base case: For $n = 0$, $(1+x)^0 = 1 \geq 1 = 1 + 0 \cdot x$ and so the equality holds.
Inductive step: Suppose now as inductive hypothesis that $(1+x)^k \geq 1 + kx$ for some non-negative integer $k$ and real numbers $x > -1$. We need to show that $(1+x)^{k+1} \geq 1 + (k+1)x$. Observe that $x > -1 \to 1 + x > 0$, and that $k \geq 0$ and $x^2 \geq 0$ (since for any real number $a$, $a^2 \geq 0$) both imply that $kx^2 \geq 0$. Thus $(1+x)^{k+1} = (1+x)(1+x)^k$ (by inductive definition). By inductive hypothesis, $(1+x)^k \geq 1 + kx \to (1+x)(1+x)^k = (1+x)^{k+1} \geq (1+kx)(1+x) = 1 + x + kx + kx^2 = 1 + (k+1)x + kx^2$ (multiply by $1 + x > 0$) $\geq 1 + (k+1)x + 0 = 1 + (k+1)x$ (since $kx^2 \geq 0$). Therefore $(1+x)^{k+1} \geq 1 + (k+1)x$ as required.
Conclusion: Hence, by induction on $n$, $(1+x)^n \geq 1 + nx$ for all non-negative integers $n$ and real numbers $x > -1$.

15. For which non-negative integer values of $n$ is $n! \geq 3^n$?

$n! \geq 3^n$ is true for $n = 0$ and all integers $n \geq 7$.
For $n = 0$, $n! = 0! = 1$ by inductive definition, and $3^n = 3^0 = 1$, and so the equality holds. Now we show that the inequality is also true for all integers $n \geq 7$.
Base case: For $n = 7$, $n! = 7! = 5040$ and $3^n = 3^7 = 2187$; and so $n! \geq 3^n$.
Inductive step: Suppose now as inductive hypothesis that $k! \geq 3^k$ for some integer $k \geq 7$. We need to show that $(k+1)! \geq 3^{k+1}$. Then $3k! \geq 3 \cdot 3^k = 3^{k+1}$ and $(k+1)! = (k+1)k!$ (by inductive definition) $\geq (7+1)k! = 8k!$ (since $k \geq 7$)$\geq 3k! \geq 3 \cdot 3^k = 3^{k+1}$ (by inductive hypothesis). Thus $(k+1)! \geq 3^{k+1}$ as required.
Conclusion: Hence, by induction on $n$, $n! \geq 3^n$ for all integers $n \geq 7$.

16. Prove by induction on $n$ that
$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1},$$
for all positive integers $n$.

Base case: For $n = 1$,

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$$

and

$$\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$$

as required.

Inductive step: Suppose now as inductive hypothesis that

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

for some positive integer $k$. We need to show that

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{(k+1)+1} = \frac{k+1}{k+2}.$$

But by inductive definition and inductive hypothesis

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2}\end{aligned}$$

as required.

Conclusion: Hence, by induction on $n$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

for all positive integers $n$.

17. For a positive integer $n$ the number $a_n$ is defined inductively by

$$a_1 = 1,$$
$$a_{k+1} = \frac{6a_k + 5}{a_k + 2}$$

for $k$ a positive integer.

Prove by induction on $n$ that, for all positive integers, (i) $a_n > 0$ and (ii) $a_n < 5$.

(i) Base case: For $n = 1$, $a_n = a_1 = 1 > 0$ as required.

Inductive step: Suppose now as inductive hypothesis that $a_k > 0$ for some positive integer $k$. We need to show that

$$a_{k+1} = \frac{6a_k + 5}{a_k + 2} > 0.$$

By inductive hypothesis, $a_k > 0 \to a_k + 2 > 0$ and $a_k > 0 \to 2a_k > a_k \to 6a_k > a_k \to 6a_k + 5 > a_k + 2 > 0$. Thus

$$\frac{6a_k + 5}{a_k + 2} > 1 > 0 \to \frac{6a_k + 5}{a_k + 2} > 0$$

as required.

<u>Conclusion:</u> Hence, by induction on $n$, $a_n > 0$ for all positive integers $n$.

(ii) <u>Base case:</u> For $n = 1$, $a_n = a_1 = 1 < 5$ as required.

<u>Inductive step:</u> Suppose now as inductive hypothesis that $a_k < 5$ for some positive integer $k$. We need to show that

$$a_{k+1} = \frac{6a_k + 5}{a_k + 2} < 5.$$

By inductive hypothesis, $a_k < 5 \to 6a_k < 5a_k + 5 \to 6a_k + 5 < 5a_k + 10 = 5(a_k + 2) \to 6a_k + 5 < 5(a_k + 2)$. Thus

$$\frac{6a_k + 5}{a_k + 2} < 5$$

as required.

<u>Conclusion:</u> Hence, by induction on $n$, $a_n < 5$ for all positive integers $n$.

18. Given a sequence of numbers $a(1), a(2), \ldots$, the number $\prod_{i=1}^{n} a(i)$ is defined inductively by

(i) $\displaystyle\prod_{i=1}^{1} \mathrm{a(i)} = a(1)$, and

(ii) $\displaystyle\prod_{i=1}^{k+1} a(i) = \left(\prod_{i=1}^{k} a(i)\right) a(k+1)$ for $k \geq 1$.

Prove that

$$\prod_{i=1}^{n} \left(1 + x^{2^{i-1}}\right) = \frac{1 - x^{2^n}}{1 - x} \text{ for } x \neq 1.$$

What happens if $x = 1$?

<u>Base case:</u> For $n = 1$,

$$\prod_{i=1}^{1} \left(1 + x^{2^{i-1}}\right) = 1 + x^{2^{1-1}} = 1 + x^{2^0} = 1 + x$$

and

$$\frac{1 - x^{2^n}}{1 - x} = \frac{1 - x^{2^1}}{1 - x} = \frac{(1 - x)(1 + x)}{1 - x} = 1 + x$$

as required.

<u>Inductive step:</u> Suppose now as inductive hypothesis that

$$\prod_{i=1}^{k} \left(1 + x^{2^{i-1}}\right) = \frac{1 - x^{2^k}}{1 - x}$$

for some integer $k \geq 1$ and for all real numbers $x \neq 1$. We need to show that

$$\prod_{i=1}^{k+1}\left(1+x^{2^{i-1}}\right)=\frac{1-x^{2^{k+1}}}{1-x}.$$

By inductive definition and inductive hypothesis,

$$\begin{aligned}\prod_{i=1}^{k+1}\left(1+x^{2^{i-1}}\right)&=\left[\prod_{i=1}^{k}\left(1+x^{2^{i-1}}\right)\right]\cdot\left(1+x^{2^{(k+1)-1}}\right)\\&=\frac{1-x^{2^k}}{1-x}\left(1+x^{2^k}\right)\\&=\frac{1-x^{2^{k+1}}}{1-x}\end{aligned}$$

as required.

<u>Conclusion:</u> Hence

$$\prod_{i=1}^{n}\left(1+x^{2^{i-1}}\right)=\frac{1-x^{2^n}}{1-x}$$

for all integers $n \geq 1$ and for all real numbers $x \neq 1$. Moreover, if $x = 1$, then the formula does not work since $1 - x = 0$ and we cannot divide by zero. However, $x^{2^{i-1}} = 1$ for all $i \geq 1$ and so

$$\prod_{i=1}^{n}\left(1+x^{2^{i-1}}\right)=2^n.$$

19. Prove that

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}$$

for integers $n \geq 2$.

<u>Base case:</u> For $n = 2$,

$$\prod_{i=2}^{2}\left(1-\frac{1}{i^2}\right)=1-\frac{1}{2^2}=\frac{3}{4}$$

and

$$\frac{n+1}{2n}=\frac{2+1}{2(2)}=\frac{3}{4}$$

as required.

<u>Inductive step:</u> Suppose now as inductive hypothesis that

$$\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)=\frac{k+1}{2k}$$

for some integer $k \geq 2$. We need to show that

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right)=\frac{(k+1)+1}{2(k+1)}=\frac{k+2}{2k+2}.$$

By inductive definition and inductive hypothesis,

$$\begin{aligned}
\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right) &= \left[\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)\right]\cdot\left(1-\frac{1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k}\cdot\left(1-\frac{1}{(k+1)^2}\right) \\
&= \frac{(k+1)[(k+1)^2-1]}{2k(k+1)^2} \\
&= \frac{(k+1)^2-1}{2k(k+1)} \\
&= \frac{k^2+2k+1-1}{2k(k+1)} \\
&= \frac{k(k+2)}{2k(k+1)} \\
&= \frac{k+2}{2k+2}
\end{aligned}$$

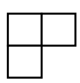
as required.

<u>Conclusion:</u> Hence

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for all integers $n \geq 2$.

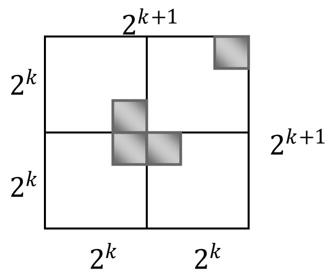
20. Prove that, for a positive integer $n$, a $2^n \times 2^n$ square grid with any one square removed can be covered using L-shaped tiles such as the one shown below.



<u>Base case:</u> For $n = 1$, a $2^1 \times 2^1$ square with one square removed can be covered by a single L-shaped tile.

<u>Inductive step:</u> Suppose now as inductive hypothesis that a $2^k \times 2^k$ square grid with any one square removed can be covered by L-shaped tiles. We need to deduce that a $2^{k+1} \times 2^{k+1}$ square grid with any one square removed can be covered using L-shaped tiles. If we divide the $2^{k+1} \times 2^{k+1}$ square grid in four equal square grids (as shown in the figure below), we obtain four $2^k \times 2^k$ square grids (observe that $2^{k+1}/2 = 2^k$). Since the $2^{k+1} \times 2^{k+1}$ square grid has one square removed, this removed square must lie in one of the four $2^k \times 2^k$ square grids (as shown by the shaded square in the corner of the figure below). The other three $2^k \times 2^k$ square grid are complete. Now from each of the complete $2^k \times 2^k$ square grids, remove the square that touches the center of the original $2^{k+1} \times 2^{k+1}$ square grid (as shown in the figure below). By induction hypothesis, all four of the $2^k \times 2^k$ square grids with one square removed can be covered using L-shaped tiles. Then, with one more L-shaped tile, we can cover the three squares touching the center of the original $2^{k+1} \times 2^{k+1}$ square grid.

Thus we can cover the original $2^{k+1} \times 2^{k+1}$ square grid with one square removed using L-shaped tiles as required.
<u>Conclusion:</u> Hence, for a positive integer $n$, a $2^n \times 2^n$ square grid with any one square removed can be covered using L-shaped tiles.




21. Suppose that $x$ is a real number such that $x + 1/x$ is an integer. Prove by induction on $n$ that $x^n + 1/x^n$ is an integer for all positive integers $n$.
[For the inductive step consider $(x^k + 1/x^k)(x + 1/x)$.]

Strong induction is used.
<u>Base case:</u> For $n = 1$, $x^n + 1/x^n = x + 1/x$ is an integer as required. Now $(x^n + 1/x^n)^2 = x^2 + 1/x^2 + 2$ is an integer since the square of an integer is an integer and thus $x^2 + 1/x^2$ is an integer (since for any integer $a$, $a - 2$ is an integer) proving the result for $n = 2$.
<u>Inductive step:</u> Suppose now as inductive hypothesis that $x^k + 1/x^k$ is an integer for all positive integers $n \leq k$ for some integer $k \geq 2$. We need to show that $x^{k+1} + 1/x^{k+1}$ is an integer. By inductive hypothesis, $(x + 1/x)(x^k + 1/x^k)$ is an integer since the product of two integers is an integer. But

$$\left(x + \frac{1}{x}\right)\left(x^k + \frac{1}{x^k}\right) = x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}}$$

and, by inductive hypothesis, $x^{k-1} + 1/x^{k-1}$ is an integer. Thus $x^{k+1} + 1/x^{k+1}$ must be an integer as required.
<u>Conclusion:</u> Hence, by induction on $n$, $x^n + 1/x^n$ is an integer for all positive integers $n$.

22. Prove that

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

for positive integers $n$ and positive real numbers $x_i$.
[it does not seem to be possible to give a direct proof of this result using induction on $n$. However it can be proved for $n = 2^m$ for $m \geq 0$ by induction on $m$. The general result now follows by proving the converse of the usual inductive step: if the result holds for $n = k + 1$, where $k$ is a positive integer, then it holds for $n = k$.]

case 1: If all the terms of the sequence $x_i$ are equal, that is $x_1 = x_2 = \cdots = x_n$, then

$$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}(nx_1) = x_1 = ({x_1}^n)^{1/n} = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

which implies that

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

as required.

case 2: If not all the terms of the sequence $x_i$ are equal. Clearly this case is only possible when $n > 1$, and it is proved by induction. First we prove the inequality when $n = 2^m$ for $m \geq 1$ and then, using this result, we deduce that the inequality is true for all positive integers $n$.

Base case: For $m = 1$, $n = 2^m = 2^1 = 2$. So we have two terms, $x_1$ and $x_2$, and since they are not equal, we have

$$x_1 \neq x_2$$
$$x_1 - x_2 \neq 0$$
$$(x_1 - x_2)^2 > 0$$
$${x_1}^2 - 2x_1x_2 + {x_2}^2 > 0$$
$${x_1}^2 + 2x_1x_2 + {x_2}^2 > 4x_1x_2$$
$$(x_1 + x_2)^2 > 4x_1x_2$$
$$\left(\frac{x_1 + x_2}{2}\right)^2 > x_1x_2$$
$$\frac{x_1 + x_2}{2} > \sqrt{x_1x_2}$$

and so

$$\frac{1}{2}\sum_{i=1}^{2} x_i \geq \left(\prod_{i=1}^{2} x_i\right)^{1/2}$$

as required.

Inductive step: Suppose now as inductive hypothesis that

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

where $n = 2^m$, for some positive integer $m$. We need to show that

$$\frac{1}{2^{m+1}}\sum_{i=1}^{2^{m+1}} x_i \geq \left(\prod_{i=1}^{2^{m+1}} x_i\right)^{1/2^{m+1}} .$$

Then by inductive hypothesis

$$\frac{1}{2^{m+1}}\sum_{i=1}^{2^{m+1}} x_i = \frac{x_1 + x_2 + \cdots + x_{2^{m+1}}}{2^{m+1}}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{x_1 + x_2 + \cdots + x_{2^{m+1}}}{2^m}\right) \\
&= \frac{1}{2}\left(\frac{x_1 + x_2 + \cdots + x_{2^m}}{2^m}\right) + \frac{1}{2}\left(\frac{x_{2^m+1} + x_{2^m+2} + \cdots + x_{2^{m+1}}}{2^m}\right) \\
&= \frac{1}{2}\left[\left(\frac{x_1 + x_2 + \cdots + x_{2^m}}{2^m}\right) + \left(\frac{x_{2^m+1} + x_{2^m+2} + \cdots + x_{2^{m+1}}}{2^m}\right)\right] \\
&\geq \frac{\sqrt[2^m]{x_1 \cdot x_2 \cdots x_{2^m}} + \sqrt[2^m]{x_{2^m+1} \cdot x_{2^m+2} \cdots x_{2^{m+1}}}}{2} \\
&\geq \sqrt{\sqrt[2^m]{x_1 \cdot x_2 \cdots x_{2^m}} \times \sqrt[2^m]{x_{2^m+1} \cdot x_{2^m+2} \cdots x_{2^{m+1}}}} \\
&= \sqrt{\sqrt[2^m]{x_1 \cdot x_2 \cdots x_{2^{m+1}}}} \\
&= \sqrt[2^{m+1}]{x_1 \cdot x_2 \cdots x_{2^{m+1}}} \\
&= \left(\prod_{i=1}^{2^{m+1}} x_i\right)^{1/2^{m+1}}
\end{aligned}
$$

as required. Hence, by induction on $m$, the result is true for all positive integers $m$. Thus the inequality is true for the natural powers of 2, that is for $n = 2,4,8,16,\ldots$

Now we proceed to prove the inequality for all positive integers $n$. If $n$ is not equal to some natural power of 2, then it is certainly less than some natural power of 2, since the sequence $2,4,8,16,\ldots,2^m,\ldots$ is unbounded above. Therefore let $m$ be some natural power of 2 that is greater than $n$. Also let

$$\frac{1}{n}\sum_{i=1}^{n} x_i = \alpha$$

and expand our list of terms such that

$$x_{n+1} = x_{n+2} = \cdots = x_m = \alpha.$$

Then

$$
\begin{aligned}
\alpha &= \frac{x_1 + x_2 + \cdots + x_n}{n} \\
&= \frac{\frac{m}{n}(x_1 + x_2 + \cdots + x_n)}{m} \\
&= \frac{\left(\frac{m}{n}+1-1\right)(x_1 + x_2 + \cdots + x_n)}{m} \\
&= \frac{x_1 + x_2 + \cdots + x_n + \left(\frac{m-n}{n}\right)(x_1 + x_2 + \cdots + x_n)}{m} \\
&= \frac{x_1 + x_2 + \cdots + x_n + (m-n)(\alpha)}{m} \\
&= \frac{x_1 + x_2 + \cdots + x_n + x_{n+1} + \cdots + x_m}{m} \\
&\geq \sqrt[m]{x_1 \cdot x_2 \cdots x_n \cdot x_{n+1} \cdots x_m} \\
&= \sqrt[m]{x_1 \cdot x_2 \cdots x_n \cdot \alpha^{m-n}}
\end{aligned}
$$

and so

$$\alpha^m \geq x_1 \cdot x_2 \cdots x_n \cdot \alpha^{m-n}$$
$$\alpha^n \geq x_1 \cdot x_2 \cdots x_n$$
$$\alpha \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$$
$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

as required. Hence, by induction on $n$, the inequality is true for all positive integers $n$ and all positive real numbers $x_i$.

23. For non-zero real numbers $x$ we may extend Definition 5.3.3 to a definition of powers $x^n$ for all integers $n$ by defining $x^{-m} = 1/x^m$ for integers $m > 0$. With these definitions prove the laws of exponents for any non-zero real numbers $x$ and $y$ and integers $m$ and $n$:
(i) $x^n y^n = (xy)^n$;
(ii) $x^{m+n} = x^m x^n$;
(iii) $(x^m)^n = x^{mn}$.
[Hint: Start from exercise 5.7.]

(i) First, we prove the result for the non-negative integers.
Base case: For $n = 0$, $x^n y^n = 1 = (xy)^n$ as required.
Inductive step: Suppose now as inductive hypothesis that $x^k y^k = (xy)^k$ for some non-negative integer $k$. We need to show that $x^{k+1}y^{k+1} = (xy)^{k+1}$. Then, by inductive definition and inductive hypothesis, $x^{k+1}y^{k+1} = \left(x \cdot x^k\right)\left(y \cdot y^k\right) = (xy)(xy)^k = (xy)^{k+1}$ as required to prove the result for $n = k + 1$.
Conclusion: Hence, by induction on $n$, $x^n y^n = (xy)^n$ for any non-zero real numbers $x$ and $y$ and non-negative integers $n$.
Now we prove the result for the non-positive integers by proving that
$$\frac{1}{x^n} \cdot \frac{1}{y^n} = \frac{1}{(xy)^n}$$
for all non-negative integers $n$. Then, by the definition of $x^{-m}$, we can conclude that the result is true for all the non-positive integers.
Base case: For $n = 0$,
$$\frac{1}{x^n} \cdot \frac{1}{y^n} = 1 = \frac{1}{(xy)^n}$$
as required.
Inductive step: Suppose now as inductive hypothesis that
$$\frac{1}{x^k} \cdot \frac{1}{y^k} = \frac{1}{(xy)^k}$$
for some non-negative integer $k$. We need to show that
$$\frac{1}{x^{k+1}} \cdot \frac{1}{y^{k+1}} = \frac{1}{(xy)^{k+1}}.$$
Then, by inductive definition and inductive hypothesis,
$$\frac{1}{x^{k+1}} \cdot \frac{1}{y^{k+1}} = \frac{1}{x \cdot x^k} \cdot \frac{1}{y \cdot y^k} = \frac{1}{(xy)(xy)^k} = \frac{1}{(xy)^{k+1}}$$

as required to prove the result for $n = k + 1$.
<u>Conclusion:</u> Hence, by induction on $n$,

$$\frac{1}{x^n} \cdot \frac{1}{y^n} = \frac{1}{(xy)^n}$$

for any non-zero real numbers $x$ and $y$ and non-negative integers $n$.
Therefore $x^n y^n = (xy)^n$ for any non-zero real numbers $x$ and $y$ and any integer $n$.

(ii) First, we prove the result for the non-negative integers.
<u>Base case:</u> For $n = 0$, $x^{m+n} = x^m = x^m x^0 = x^m x^n$ as required.
<u>Inductive step:</u> Suppose now as inductive hypothesis that $x^{m+k} = x^m x^k$ for some non-negative integer $k$. We need to show that $x^{m+k+1} = x^m x^{k+1}$. Then, by inductive definition and inductive hypothesis, $x^{m+k+1} = xx^{m+k} = xx^m x^k = x^m xx^k = x^m x^{k+1}$ as required to prove the result for $n = k + 1$.
<u>Conclusion:</u> Hence, by induction on $n$, $x^{m+n} = x^m x^n$ for any non-zero real number $x$ and non-negative integers $m$ and $n$.
Now we prove the result for the non-positive integers by proving that

$$\frac{1}{x^{m+n}} = \frac{1}{x^m x^n}$$

for all non-negative integers $m$ and $n$. Then, by the definition of $x^{-m}$, we can conclude that the result is true for all the non-positive integers.
<u>Base case:</u> For $n = 0$,

$$\frac{1}{x^{m+n}} = \frac{1}{x^m} = \frac{1}{x^m x^0} = \frac{1}{x^m x^n}$$

as required.
<u>Inductive step:</u> Suppose now as inductive hypothesis that

$$\frac{1}{x^{m+k}} = \frac{1}{x^m x^k}$$

for some non-negative integer $k$. We need to show that

$$\frac{1}{x^{m+k+1}} = \frac{1}{x^m x^{k+1}}.$$

Then, by inductive definition and inductive hypothesis,

$$\frac{1}{x^{m+k+1}} = \frac{1}{xx^{m+k}} = \frac{1}{xx^m x^k} = \frac{1}{x^m xx^k} = \frac{1}{x^m x^{k+1}}$$

as required to prove the result for $n = k + 1$.
<u>Conclusion:</u> Hence, by induction on $n$,

$$\frac{1}{x^{m+n}} = \frac{1}{x^m x^n}$$

for any non-zero real number $x$ and non-negative integers $m$ and $n$.
Therefore $x^{m+n} = x^m x^n$ for any non-zero real number $x$ and any integers $m$ and $n$.

(iii) First, we prove the result for the non-negative integers.
<u>Base case:</u> For $n = 0$, $(x^m)^n = 1 = x^0 = x^{mn}$ as required.

Inductive step: Suppose now as inductive hypothesis that $(x^m)^k = x^{mk}$ for some non-negative integer $k$. We need to show that $(x^m)^{k+1} = x^{m(k+1)}$. Then, by inductive definition and inductive hypothesis, $(x^m)^{k+1} = (x^m)(x^m)^k = x^m x^{mk} = x^{m+mk}$ (by part (ii) of the problem) $= x^{m(k+1)}$ as required to prove the result for $n = k + 1$.
Conclusion: Hence, by induction on $n$, $(x^m)^n = x^{mn}$ for any non-zero real number $x$ and non-negative integers $m$ and $n$.
Now we prove the result for the non-positive integers by proving that

$$\frac{1}{(x^m)^n} = \frac{1}{x^{mn}}$$

for all non-negative integers $n$. Then, by the definition of $x^{-m}$, we can conclude that the result is true for all the non-positive integers.
Base case: For $n = 0$,

$$\frac{1}{(x^m)^n} = 1 = \frac{1}{x^0} = \frac{1}{x^{mn}}$$

as required.
Inductive step: Suppose now as inductive hypothesis that

$$\frac{1}{(x^m)^k} = \frac{1}{x^{mk}}$$

for some non-negative integer $k$. We need to show that

$$\frac{1}{(x^m)^{k+1}} = \frac{1}{x^{m(k+1)}}.$$

Then, by inductive definition, inductive hypothesis, and part (ii) of the problem

$$\frac{1}{(x^m)^{k+1}} = \frac{1}{(x^m)(x^m)^k} = \frac{1}{x^m x^{mk}} = \frac{1}{x^{m+mk}} = \frac{1}{x^{m(k+1)}}$$

as required to prove the result for $n = k + 1$.
Conclusion: Hence, by induction on $n$,

$$\frac{1}{(x^m)^n} = \frac{1}{x^{mn}}$$

for any non-zero real number $x$ and non-negative integers $m$ and $n$.
Therefore $(x^m)^n = x^{mn}$ for any non-zero real number $x$ and any integers $m$ and $n$.

24. Fibonacci's rabbit problem may be stated as follows:
How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which become productive from the second month on?
Assuming that no rabbits die, express the number after $n$ months as a Fibonacci number and hence answer the problem. Using a calculator and the Binnet formula (Proposition 5.4.3) find the number after three years.

The $n$th Fibonacci number is given by the following formula:

$$u_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ and $n$ is the number of months. Thus after one year (12 months) there are

$$u_{12} = \frac{\alpha^{12} - \beta^{12}}{\sqrt{5}} = 144 \text{ rabbits}$$

and after three years (36 months) there are

$$u_{36} = \frac{\alpha^{36} - \beta^{36}}{\sqrt{5}} = 14930352 \text{ rabbits}$$

25. Let $u_n$ be the $n$th Fibonacci number (Definition 5.4.2). Prove, by induction on $n$ (Without using the Binnet formula Proposition 5.4.3), that

$$u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$$

for all positive integers $m$ and $n$.

Deduce, again using induction on $n$, that $u_m$ divides $u_{mn}$.

Strong induction is used.

Base case: For $n = 1$, $u_{m+n} = u_{m+1} = u_{m-1}u_1 + u_m u_2 = u_{m-1} + u_m$ (since, by the inductive definition of the Fibonacci sequence, $u_1 = u_2 = 1$). For $n = 2$, $u_{m+n} = u_{m+2} = u_{m-1}u_2 + u_m u_3 = u_{m-1} + 2u_m$ (since $u_3 = u_1 + u_2 = 1 + 1 = 2$). Note that by the first case $u_{m-1} = u_{m+1} - u_m$. Then $u_{m+2} = u_{m+1} - u_m + 2u_m = u_{m+1} + u_m$ as required.

Inductive step: Suppose now as inductive hypothesis that $u_{m+k} = u_{m-1}u_k + u_m u_{k+1}$ for all positive integers $n \le k$ for some positive integer $k \ge 2$. We need to show that $u_{m+k+1} = u_{m-1}u_{k+1} + u_m u_{k+2}$. Then, by the inductive definition of the Fibonacci sequence, $u_{m+k+1} = u_{m+k} + u_{m+k-1}$ and by inductive hypothesis

$$\begin{aligned} u_{m+k+1} &= u_{m-1}u_k + u_m u_{k+1} + u_{m-1}u_{k-1} + u_m u_k \\ &= u_{m-1}(u_k + u_{k-1}) + u_m(u_{k+1} + u_k) \\ &= u_{m-1}(u_{k+1}) + u_m(u_{k+2}) \\ &= u_{m-1}u_{k+1} + u_m u_{k+2} \end{aligned}$$

as required.

Conclusion: Hence, by induction on $n$, $u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$ for all positive integers $m$ and $n$.

$u_m$ divides $u_{mn}$ means that $u_{mn} = u_m q$ for some integer $q$.

Base case: For $n = 1$, $u_{mn} = u_m$ and so $u_m$ divides $u_{mn}$ as required.

Inductive step: Suppose now as inductive hypothesis that there exists some integer $q$ such that $u_{mk} = u_m q$ for some positive integers $m$ and $k$. We need to show that $u_{m(k+1)} = u_m p$ for some integer $p$. Observe that all the Fibonacci numbers are integers since every number is the sum of the previous two numbers, which in turn are integers. Then, by the first part of the problem and by inductive hypothesis,

$$\begin{aligned} u_{m(k+1)} &= u_{mk+m} \\ &= u_{mk-1}u_m + u_{mk}u_{m+1} \\ &= u_{mk-1}u_m + u_m q u_{m+1} \end{aligned}$$

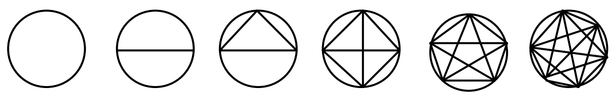
$$= u_m(u_{mk-1} + qu_{m+1})$$
$$= u_m p$$

where $p = u_{mk-1} + qu_{m+1}$ is an integer and so $u_m$ divides $u_{m(k+1)}$ as required.
Conclusion: Hence, by induction on $n$, $u_m$ divides $u_{mn}$ for all positive integers $m$ and $n$.

26. Suppose that $n$ points on a circle are all joined in pairs. The points are positioned so that no three joining lines are concurrent in the interior of the circle. Let $a_n$ be the number of regions into which the interior of the circle is divided. Draw diagrams to find $a_n$ for $n \leq 6$.
Prove that $a_n$ is given by the following formula.

$$\begin{aligned} a_n &= n + C(n-1,2) + C(n-1,3) + C(n-1,4) \\ &= 1 + n(n-1)(n^2 - 5n + 18)/24. \end{aligned}$$

The following are the drawings corresponding to $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, and $a_6$.



For $a_6$, it is not feasible to show that the center is not the intersection of three lines since we are working with small diagrams. However note that the points on the circle do not form a regular hexagon circumscribed about a circle for otherwise we would obtain three lines concurrent at the center of the circle. Thus there is another region (not visible in such diagram) at the center of the figure.
Base case: For $n = 1$, we can see that the interior of the circle is divided into one region and $a_n = n + C(n-1,2) + C(n-1,3) + C(n-1,4) = 1 = 1 + n(n-1)(n^2 - 5n + 18)/24$ as required.
Inductive step: Suppose now as inductive hypothesis that $a_k = 1 + k(k-1)(k^2 - 5k + 18)/24$ for some positive integer $k$. We need to show that $a_{k+1} = 1 + k(k+1)[(k+1)^2 - 5(k+1) + 18]/24$. Then by definition,

$$\begin{aligned} a_{k+1} &= k + 1 + C(k,2) + C(k,3) + C(k,4) \\ &= k + 1 + \frac{k!}{2!\,(k-2)!} + \frac{k!}{3!\,(k-3)!} + \frac{k!}{4!\,(k-4)!} \\ &= k + 1 + \frac{1}{2}k(k-1) + \frac{1}{6}k(k-1)(k-2) + \frac{1}{24}k(k-1)(k-2)(k-3) \\ &= 1 + \frac{7}{12}k + \frac{11}{24}k^2 - \frac{1}{12}k^3 + \frac{1}{24}k^4 \\ &= 1 + \frac{k(14 + 11k - 2k^2 + k^3)}{24} \\ &= 1 + \frac{k(k+1)(k^2 - 3k + 14)}{24} \\ &= 1 + \frac{k(k+1)[(k+1)^2 - 5(k+1) + 18]}{24} \end{aligned}$$

as required to prove the result for $n = k + 1$.

Conclusion: Hence, by induction on $n$, $a_n = 1 + n(n-1)\,(n^2 - 5n + 18)/24$ for all positive integers $n$.